$\mathbb{A}^1$-enumerative geometry via $\mathbb{A}^1$-degree

(Kass-Wickelgren)

Motivation from classical topology:

$$\deg : [S^n, S^n] \longrightarrow \mathbb{Z} \qquad \begin{array}{ccc} H_n(S^n) & \to & H_n(S^n) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z} & & \mathbb{Z} \end{array}$$

$f : S^n \to S^n$ $\quad p \in S^n$ regular value

$f^{-1}(p) = \{q_1, \dots, q_m\}$

then $\quad \deg f = \sum_{q \in f^{-1}(p)} \deg_q f$

local degree

V small ball around p $\qquad f^{-1}(p) \cap U$

U —— " —— q $\qquad = \{q\}$

$$\bar{f} : S^n \simeq U/\partial U \longrightarrow V/\partial V \simeq S^n$$

$$\begin{array}{ccc} U/\partial U & & V/\partial V \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ U/U - \{q\} & & V/V - \{p\} \end{array}$$

$\deg_q f := \deg \bar{f} \in \{\pm 1\}$ since $\bar{f}$ homeo formula from differential topology

$$\begin{array}{ccccc} T_q f : & T_q S^n & \to & T_p S^n \\ \shortparallel & \shortparallel & & \shortparallel \\ (f_1, \dots, f_n) & \mathbb{R}^n & & \mathbb{R}^n \end{array}$$

$$J(q) := \det \frac{\partial f_i}{\partial x_j}$$

Then $\deg_q f = \begin{cases} +1 & \text{if } J(q) > 0 \\ -1 & \text{if } J(q) < 0 \end{cases}$

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generalizes to degree of maps btw smooth oriented n-mflds

$$f : \underset{\text{compact}}{M} \longrightarrow \underset{\text{connected}}{N}$$

check definition

$\deg f := f_* [M]$

$= \text{sum of local degrees}$

$\operatorname{rk} E = \dim X$

$$\begin{array}{ll} \text{orientable} & E \\ & \downarrow \uparrow \\ \text{orientable} & X \ \text{compact smooth} \end{array}$$

$e(E) =$ sum of local degree of a generic section

want to do this over any field k
not only $\mathbb{R}$

htpy theory on
smooth schemes/k

Tool: $\mathbb{A}^1$-homotopy theory
and Morel's $\mathbb{A}^1$-degree

$$[\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k]_{\mathbb{A}^1} \to GW(k)$$

Note that $\mathbb{P}^n/\mathbb{P}^{n-1}(\mathbb{R}) \simeq S^n$

Need: · quotients
· $\mathbb{A}^1$-htpy classes
· $GW(k)$

Crash course in $\mathbb{A}^1$-homotopy theory

Start with Sm_k = smooth schemes $/k$
(separated of finite type)

$$Sm_k \xrightarrow{\text{Yoneda}} sPre(Sm_k) \qquad K \mapsto (U \mapsto K)$$
$$X \mapsto Map(-,X) \text{ discrete sSet} \qquad \text{constant presheaf}$$

closed under finite limits and colimits
$\Rightarrow$ can make sense of

$$\operatorname{colim}\begin{pmatrix} \mathbb{P}^{n-1}_k & \to & \mathbb{P}^n_k \\ \downarrow & & \\ * & & \end{pmatrix} = \mathbb{P}^n_k / \mathbb{P}^{n-1}_k$$

$sPre(Sm_k)$ = simplicial model cat
or ∞-cat

$\swarrow$ has notion of weak equivalence

$\searrow$ has an associated homotopy category

Bousfield localization imposes additional weak equivalences

$$Sm_k \to sPre(Sm_k) \xrightarrow[L_{Nis}]{} Sh_k \xrightarrow[L_{\mathbb{A}^1}]{} Spc_k$$

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl, dashed] \\
U \arrow[r] \arrow[ur, dashed, "P"] & X
\end{tikzcd}$$

$X \times \mathbb{A}^1 \to X$ is weak eq

$$[\ ,\ ]_{\mathbb{A}^1} = \text{maps in } ho(Spc_k)$$

↑ htpy category

Morel's degree:

$$[\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n/\mathbb{P}^{n-1}_k]_{\mathbb{A}^1} \longrightarrow GW(k)$$

↑

iso for $n>1$

epi for $n=1$

$GW(k)$ = Grothendieck-Witt ring of k
= group completion of semi-ring of isometry classes of non-degenerate bilinear symmetric forms

generators: $\langle a \rangle \quad a \in k^*$

$\langle a \rangle = (x,y) \mapsto axy$

relations: 1) $\langle a \rangle = \langle ab^2 \rangle$

2) $\langle a \rangle \langle b \rangle = \langle ab \rangle$

3) $\langle a \rangle + \langle b \rangle = \langle ab(a+b) \rangle + \langle a+b \rangle$

(4) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$)

Ex: · $GW(\mathbb{C}) \underset{rk}{\cong} \mathbb{Z}$

· $GW(\mathbb{R}) \underset{(rk, sgn)}{\cong} \mathbb{Z} \times \mathbb{Z}$

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$

sgn = # 1 − # −1

· $GW(\mathbb{F}_q) \cong \mathbb{Z} \times \mathbb{F}_q^* / (\mathbb{F}_q^*)^2$

$\uparrow$

(rk, disc)

$\shortparallel$

det matrix

$\mathbb{A}^1$-local degree

$f : \mathbb{P}^n/\mathbb{P}^{n-1} \to \mathbb{P}^n/\mathbb{P}^{n-1}$

$\deg : [S^n, S^n] \to \mathbb{Z} \rightsquigarrow \deg^{\mathbb{A}^1} : [\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k]_{\mathbb{A}^1} \to GW(k)$

$f : S^n \to S^n \qquad p \in S^n$ regular value

$f^{-1}(p) = \{q_1, \dots, q_m\}$

then $\qquad \deg f = \sum_{q \in f^{-1}(p)} \deg_q f$

local degree

V ~~small ball~~ around $p \qquad f^{-1}(p) \cap U = \{q\}$

U — " — q

$\bar{f} : S^n \simeq U/\partial U \to V/\partial V \simeq S^n$

$\mathbb{P}^n/\mathbb{P}^{n-1} \simeq \mathbb{A}^n/\mathbb{A}^n - 0 \underset{\text{coord}}{\overset{\text{Nis}}{\simeq}} U/U - \{q\} \qquad V/V - \{p\} \underset{\text{coord}}{\overset{\text{Nis}}{\simeq}} \mathbb{A}^n/\mathbb{A}^n - 0 \simeq \mathbb{P}^n/\mathbb{P}^{n-1}$

$\deg_q f := \deg \bar{f} \in \{\pm 1\}$ since $\bar{f}$ homeo

$\rightsquigarrow \bar{f} : \mathbb{A}^n/\mathbb{A}^n - \{q\} \to \mathbb{A}^n/\mathbb{A}^n - \{p\}$

$\simeq \mathbb{P}^n/\mathbb{P}^{n-1} \qquad \simeq \mathbb{P}^n/\mathbb{P}^{n-1}$

$$\deg_q f := \deg \bar{f}$$

Computation:

$$T_q f : T_q S^n \to T_p S^n \rightsquigarrow \mathbb{A}^n \to \mathbb{A}^n$$

$$\begin{array}{ccc} T_q f & T_q S^n & T_p S^n \\ \shortparallel & \shortparallel & \shortparallel \\ (f_1, \dots, f_n) & \mathbb{R}^n & \mathbb{R}^n \end{array}$$

$$J(q) := \det \frac{\partial f_i}{\partial x_j}$$

Then $\deg_q f = \begin{cases} +1 & \text{if } J(q) > 0 \\ -1 & \text{if } J(q) < 0 \end{cases}$

$$\deg_q f := \langle J(q) \rangle$$

$$\text{if } J(q) \neq 0$$

If q not defined over k

then $\mathrm{Tr}_{k(q)/k}(\langle J(q) \rangle)$

$$\mathrm{Tr}_{L/k} : GW(L) \longrightarrow GW(k)$$

L/k $\quad V \times V \xrightarrow{\beta} L \longmapsto \left(V \times V \xrightarrow{\beta} L \xrightarrow{\mathrm{Tr}_{L/k}} k \right)$

Counting lines

The $\mathbb{A}^1$-Euler number

$\pi : \mathcal{E} \to X$ $\overset{\text{VB}}{\text{rk}\ r}$ $\dim X = r$

Def $\pi : \mathcal{E} \to X$ is *relatively oriented* if $\mathrm{Hom}(\det TX, \det E) \cong L^{\otimes 2}$

take a general section $\sigma : X \to \mathcal{E}$ has finite number of zeros

Can choose local "Nisnevich coordinate" around every zero, compatible with relative orientation ie

$$\det TX|_u \to \det E|_u$$

dist elemt $\mapsto$ dist elemt $\mapsto$ square $1 \otimes 1$

Def The $\mathbb{A}^1$-Euler number $e(\mathcal{E}, \sigma)$

$\pi: \mathcal{E} \to X$ is

$$\sum_{q \in \sigma^{-1}(0)} \mathrm{ind}_q$$

local degrees

Fact (Kass-Wickelgren, Bachmann-Wickelgren)

This does not depend on the chosen section

Application:

1) Counting lines on cubic surfaces

$X = \{f = 0\} \subseteq \mathbb{P}^3$ ← homogeneous of degree 3

$\rightsquigarrow$ section $\sigma_f : Gr(2,4) \to Sym^3 S^*$ by restriction

$Gr(2,4)$ = lines in $\mathbb{P}^3$, S = tautological bundle

lines on X = zeros of section

$\dim Gr(2,4) = 4$

$\mathrm{rk}\, Sym^3 S^* = 4$

Macaulay 2 $\Rightarrow$ rk 27, sgn 3, disc 1

$\rightsquigarrow 15\langle 1\rangle + 12\langle -1\rangle \in GW(k)$

2) Counting singular elements on a pencil of degree d surfaces

Falls es zeitlich passt

$F_t = F_0 t_0 + F_1 t_1 = 0 \subseteq \mathbb{P}^3 \times \mathbb{P}^1$

pencil of degree d surfaces

an elmt of pencil is singular $\iff \exists$ pt where

$$\frac{\partial F_t}{\partial x_0}, \dots, \frac{\partial F_t}{\partial x_3}$$

vanish

$$\rightsquigarrow \quad \begin{tikzcd} \bigoplus_4 \pi_1^* \mathcal{O}_{\mathbb{P}^3}(d-1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1) \arrow[d] \\ \mathbb{P}^3 \times \mathbb{P}^1 \arrow[u, bend right, "G_{F_t}"'] \end{tikzcd} \qquad G_{F_t} = \frac{\partial F_t}{\partial x_0}, \dots, \frac{\partial F_t}{\partial x_3}$$

$$\begin{tikzcd} & \mathbb{P}^3 \times \mathbb{P}^1 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^3 & & \mathbb{P}^1 \end{tikzcd}$$

3) Lines on Quintic 3-folds

$= e\left(\mathrm{Sym}^5 S^* \to Gr(2,5)\right)$

Problem: too hard for my computer

Thm (Albano-Katz): There are 2875 distinguished complex lines on X that deform with

$$X_t = \{ F + t F_1 + t^2 F_2 + \ldots = 0 \} \subseteq \mathbb{P}^4$$

1) Lines in the intersection of 2 components $\ell = (s : -\zeta s : t : -\zeta' t : 0)$ deform with multiplicity 5

2) in each component there are 10 lines which deform with multiplicity 2

In total $50 \cdot 10 \cdot 2 + 375 \cdot 5 = 2875$

$\sim \frac{\binom{5}{2}\binom{3}{2}}{2} \cdot 25$

of a distinguished line

Compute $\sum \mathrm{ind}(\ell_t) \in GW(k((t)))$

$\rightsquigarrow \; 50 \cdot 10 \cdot \mathbb{H} + 15(2\mathbb{H} + \langle 1 \rangle) + 90\, \mathrm{Tr}_{k(\sqrt{5})/k}(2\mathbb{H} + \langle 1 \rangle)$

$= 1340\,\mathbb{H} + 90(\langle 1 \rangle + \langle -5 \rangle) + 15\langle 1 \rangle$

$\sim \; 1445\langle 1 \rangle + 1430\langle -1 \rangle$ for char $k \neq 5$

Q: What geometric information does $\mathrm{ind}(\ell)$ give?

cubic surfaces (Segre over $\mathbb{R}$, Kass-Wickelgren over k, char $k \neq 2$)

$\ell \subseteq X \subseteq \mathbb{P}^3$ cubic surface

Gauß map

$$\ell \cong \mathbb{P}^1 \xrightarrow{\deg 2} \mathbb{P}^1 = \begin{array}{l}\text{2-planes}\\ \text{in } \mathbb{P}^3\\ \text{containing } \ell\end{array}$$

$$p \mapsto T_pX$$

for a $p \in \ell$ $\exists!\ q \in \ell$ with $T_pX = T_qX$

$\rightsquigarrow$ involution $i : \ell \to \ell$ sending p to q

fixed pts of i are defined over $k(\sqrt{\alpha})$ $\alpha \in k^\times/(k^\times)^2$

Call $\langle \alpha \rangle \in GW(k)$ the type of ℓ

Thm: $\mathrm{Type}(\ell) = \mathrm{ind}(\ell) \in GW(k)$

Ex: over $\mathbb{R}$ there are 2 types

Quintic 3-folds (Finashin–Kharlamov, P.)

over $\mathbb{R}$ over k

$\ell \subseteq X \subseteq \mathbb{P}^4$

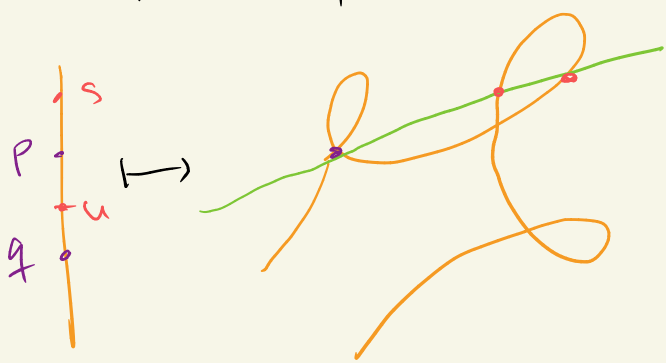
quintic 3-fold

Gauß map

$\ell \cong \mathbb{P}^1 \xrightarrow{\deg 4} \mathbb{P}^2 =$ 3-planes in $\mathbb{P}^4$ containing ℓ

$p \longmapsto T_p X$

- $\exists$ 3 pairs of pts on ℓ with the same tangent space in X



- Let p, q be such a pair and let $T = T_p X = T_q X$

then to $s \in \ell$ $\exists!$ $u \in \ell$

st $T \cap T_s X = T \cap T_u X$

$\rightsquigarrow$ 3 involutions with fixed pts defined over

$F_1(\sqrt{\alpha_1}), F_2(\sqrt{\alpha_2}), F_2(\sqrt{\alpha_3})$

Def

$\mathrm{Type}(\ell)$

$:= \prod \langle N_{F_i/k} \alpha_i \rangle$

Galois orbits $\in GW(k)$

Thm (P): $\mathrm{Type}(\ell) = \mathrm{ind}(\ell)$

So $$\sum_{\ell \subseteq X} \mathrm{Tr}_{k(\ell)/k}(\mathrm{Type}(\ell)) = 15\langle 1\rangle + 12\langle -1\rangle \in GW(k)$$

X ← cubic surface

and

$$\sum_{\ell \subseteq X} \mathrm{Tr}_{k(\ell)/k}(\mathrm{Type}(\ell)) = 1445\langle 1\rangle + 1430\langle -1\rangle \in GW(k)$$

X ← quintic 3-fold